Working Paper

EQUILIBRIUM SELECTION IN COMMON-VALUE SECOND-PRICE AUCTIONS

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This paper considers the problem of equilibrium selection in a common-value second-price auction with two bidders. We show that for each ex post equilibrium in continuous and undominated strategies of the common-value auction, a sequence of "almost common-value" auctions can be constructed such that each of them possesses a unique undominated and continuous equilibrium and the corresponding sequence of equilibria converges to that ex post equilibrium. Therefore, no equilibrium selection of this model based on perturbations seems to be more convincing than others.

1. INTRODUCTION

The well-known linkage principle (Milgrom and Weber (1982)) in auction theory states that the expected revenue in the symmetric equilibrium of a second-price auction is no less than the expected revenue from a first-price auction. However, Milgrom (1981) uses a simple example to illustrate that there could be a continuum of *asymmetric* equilibria in common-value second-price auctions, which are not revenue-equivalent. Departing from the symmetric equilibrium, it is easy to see that seller's revenue in an asymmetric equilibrium can be very low. Thus, the presence of multiple equilibria generates difficulties for revenue comparisons in common-value auctions. While in the original work, Milgrom called these asymmetric equilibria "strange", a subsequent study by Klemperer (1989) suggests that the asymmetric equilibria may be the only "reasonable" ones in the sense that by giving a slight advantage to one bidder, almost all equilibria are "extreme" as the advantageous bidder wins the auction with probability one in any undominated and continuous equilibrium. Therefore, there seems to be no obvious reason to favor the symmetric equilibrium over the asymmetric ones.

In response to this multiplicity problem, various studies have been devoted to selecting a particular equilibrium in the second-price auction by perturbing the model slightly in different ways[1]. Previous work can be summarized into three categories according to the sources of perturbation considered.

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[1] There is also an alternative approach to equilibrium selection based on iterated elimination of dominated strategies. Harstad and Levin (1985), for example, select the symmetric equilibrium in the symmetric model introduced by Milgrom and Weber (1982) using iterated elimination of dominated strategies under some restrictions on signal distributions.

First, Parreiras (2006) perturbs the second-price auction format to a hybrid auction involving the winner paying the highest bid with a small probability and the second highest bid with the complementary probability. He shows that the hybrid auction generates at least as much revenue as the first-price auction when signals are affiliated, thereby providing a justification for the linkage principle.

Second, Abraham et al. (2012) define a notion of tremble-robust equilibrium based on the idea that there is a small probability that an additional bidder may be present in the auction[2] and draws her bid according to a predetermined smooth distribution. In a model with asymmetrically-informed bidders, they select the equilibrium that generates the lowest revenue for the seller. They also identify a sufficient condition for uniqueness of tremble-robust equilibria in models with n bidders. However, different from most studies following Milgrom and Weber (1982), they consider private signals with finite support and therefore their result cannot be applied to the standard auction models with continuous signals.

Papers in the third category consider payoff perturbations. Cheng and Tan (2010) focus on the symmetric equilibrium in the second-price auction and study the revenue implications of auctions with different formats. In the working paper version (Cheng and Tan (2008)), they provide a justification of the symmetric equilibrium by adding a small private-value component to the common value model. Larson (2009) also considers private-value perturbations of the common-value auction with two bidders. He shows that asymmetric perturbations lead to selections of asymmetric equilibria, but different from our analysis, he assumes that the private-value component is independent of the common-value signals and puts restrictions on the common value component and signal distributions.

In contrast to the previous literature, this paper provides a general analysis of equilibrium selection in common-value second-price auctions. For such auctions, we provide a negative conclusion to the approach of equilibrium selection based on payoff perturbations. In particular, we show that, both in the classical pure common-value auctions and in auctions with asymmetrically informed bidders, every increasing and continuous equilibrium can be selected by perturbing bidders' valuations in a certain manner. An implication of the result is that symmetric equilibrium can only survive under a symmetric perturbation of payoffs. In the case with more than two bidders, similar results hold in both second-price auctions and English auctions.[3]

While the main results apply to equilibria in monotone and continuous strategies, we also identify a class of equilibria in discontinuous and undominated strategies which may not even be monotone in the second-price common-value auction.[4] How-

[2]Equilibrium selection via the introduction of a noisy bidder was first considered by Hashimoto (2010) in a complete information generalized second-price auction.

[3]In the two-bidder case, the second-price auction and the English auction are equivalent.

[4]Birulin (2003) points out that there exist undominated *ex post* equilibria in discontinuous strate-

ever, we show that all those discontinuous equilibria are fragile to the introduction of a noisy bid. In contrast, all equilibria in continuous and undominated strategies are again robust to this perturbation, thereby justifying our focus on the continuous equilibria.

The rest of the paper is organized as follows. Section 2 describes the common value auction and explains the multiplicity of equilibria. Section 3 investigates equilibria in perturbed auctions and their implications for equilibrium selection. Section 4 extends the perturbation idea to study a second-price auction with asymmetrically informed bidders, a game that also has a multiplicity of equilibria. Section 5 considers equilibria in discontinuous strategies.

2. A COMMON-VALUE SECOND-PRICE AUCTION WITH TWO BIDDERS

Consider a pure common-value auction with two bidders.[5] There is a single object for sale and two risk-neutral bidders compete for the object via a sealed-bid second price auction. The value of the object V is the same to both bidders. Prior to submitting bids, each bidder receives a private signal that partially reveals the value of the object. For each $i = 1, 2$, let $\tilde{s}_i$ denote bidder i's private signal. Assume $(\tilde{s}_1, \tilde{s}_2)$ is drawn according to the cumulative distribution function F with support $[0, 1] \times [0, 1]$. For each $i, j \in \{1, 2\}$ and $i \neq j$, let $F_i(\cdot|s_j)$ denote the distribution of s_i conditional on bidder j's signal realization s_j. Assume that $F_i(\cdot|s_j)$ admits a density function $f_i(\cdot|s_j)$ which is strictly positive on $[0, 1]$. The expected value of the object conditional on the signal pair (s_1, s_2) is given by $\mathbb{E}[V|s_1, s_2] = v(s_1, s_2)$. Finally, assume that v is continuously differentiable and strictly increasing in each s_i.

Since there are two bidders, this model is equivalent to an English (open ascending-price) auction. It is well-known that this pure common-value auction has multiple equilibria.[6] The following class of undominated *ex post* equilibria is identified by Milgrom (1981).

LEMMA 2.1 *For every strictly increasing and onto function $h : [0, 1] \to [0, 1]$, the strategy profile $\beta_1(s_1) = v(s_1, h^{-1}(s_1))$ and $\beta_2(s_2) = v(h(s_2), s_2)$ is an* ex post *equilibrium that is undominated. Furthermore, all undominated* ex post *equilibria in continuous strategies are of this form.*

PROOF: See Milgrom (1981) and Bikhchandani and Riley (1991). □

gies when the auction admits an efficient *ex post* equilibrium.

[5]The case with three or more bidders is discussed in Remark 3.5 of Section 3.

[6]Milgrom (1981) first pointed out the multiplicity of ex post equilibria in common-value second-price auction. Bikhchandani and Riley (1991) argue that there is a much larger class of perfect Bayesian equilibria in English auctions with more than two bidders.

Note that the seller's revenue in an asymmetric equilibrium can be very low. For example, consider the function $h(s) = s^{\alpha}$ where α is a constant. For large α, bidder 1's bids are close to $v(s_1, 0)$ with high probability. Since the losing bid determines revenue in a second-price auction, seller's expected revenue is close to $\mathbb{E}[v(s_1, 0)]$ in this asymmetric equilibrium.

Unlike prior work that selected a particular equilibrium in second-price auctions (especially the symmetric equilibrium), in the next section we obtain a negative answer to equilibrium selection based on perturbations. Our results suggest that all these asymmetric equilibria are equally convincing.

3. EQUILIBRIUM SELECTION BY PRIVATE-VALUE PERTURBATIONS

Consider the following class of "almost common-value" second-price auctions. Let $\mathcal{H}$ denote the collection of all strictly increasing and continuous functions that map $[0, 1]$ onto $[0, 1]$. For each $h \in \mathcal{H}$, define the corresponding second-price auction $\Gamma^{\varepsilon,h}$ by perturbing the ex post payoff functions of both bidders to

$$\tilde{v}_1^{\varepsilon}(s_1, s_2) = \varepsilon s_1 + (1 - \varepsilon)v(s_1, s_2)$$
$$\tilde{v}_2^{\varepsilon}(s_1, s_2) = \varepsilon h(s_2) + (1 - \varepsilon)v(s_1, s_2)$$

where $\varepsilon > 0$ is a constant.

Given bidder 2's monotone bidding function β_2, bidder 1 with signal s_1 will bid b in order to maximize

$$\int_0^{\beta_2^{-1}(b)} [\varepsilon s_1 + (1 - \varepsilon)v(s_1, s_2) - \beta_2(s_2)]\, f_2(s_2|s_1) ds_2.$$

The corresponding first order condition is

$$\left[\varepsilon s_1 + (1 - \varepsilon)v(s_1, \beta_2^{-1}(b)) - b\right] f_2(\beta_2^{-1}(b)|s_1)\beta_2^{-1\prime}(b) = 0,$$

substituting b with the bid $\beta_1(s_1)$ implies

$$\varepsilon s_1 + (1 - \varepsilon)v(s_1, \beta_2^{-1}(\beta_1(s_1))) - \beta_1(s_1) = 0. \tag{1}$$

Similarly, given bidder 1's bidding function β_1, bidder 2 with signal s_2 chooses b to maximize

$$\int_0^{\beta_1^{-1}(b)} [\varepsilon h(s_2) + (1 - \varepsilon)v(s_1, s_2) - \beta_1(s_1)]\, f_1(s_1|s_2) ds_1.$$

The first order condition gives

$$\varepsilon h(s_2) + (1 - \varepsilon)v(\beta_1^{-1}(\beta_2(s_2)), s_2) - \beta_2(s_2) = 0. \tag{2}$$

Consider any bid $b = \beta_1(s_1)$ for some $s_1 \in [0,1]$. If there is some $s_2 \in [0,1]$ such that $\beta_2(s_2) = b$, then $s_1 = \beta_1^{-1}(\beta_2(s_2))$ and $s_2 = \beta_2^{-1}(\beta_1(s_1))$. From (1) and (2), it follows that

$$\varepsilon s_1 = b - (1-\varepsilon)v(s_1, s_2) = \varepsilon h(s_2).$$

Thus, a tie happens whenever the signal pair (s_1, s_2) is such that $s_1 = h(s_2)$. Moreover, β_1 and β_2 satisfy

$$\beta_2^{-1}(\beta_1(s_1)) = h^{-1}(s_1), \quad \forall s_1 \in [0, \bar{s}_1], \tag{3}$$

$$\beta_1^{-1}(\beta_2(s_2)) = h(s_2), \quad \forall s_2 \in [0, \bar{s}_2]. \tag{4}$$

Therefore, by (1)–(4), an equilibrium $\beta^\varepsilon = (\beta_1^\varepsilon, \beta_2^\varepsilon)$ of the perturbed second-price auction $\Gamma^{\varepsilon,h}$ must satisfy

$$\beta_1^\varepsilon(s_1) = \varepsilon s_1 + (1-\varepsilon)v(s_1, h^{-1}(s_1)), \tag{5}$$

$$\beta_2^\varepsilon(s_2) = \varepsilon h(s_2) + (1-\varepsilon)v(h(s_2), s_2). \tag{6}$$

Note that the private-value components also enter both bidders' bidding functions. This follows from the fact that the price paid by the winning bidder does not depend on her own bid in second price auctions. Existence and uniqueness of a continuous equilibrium in the perturbed auction follow directly from the analysis above. In fact, the next result also establishes that the equilibrium is *ex post*.

PROPOSITION 3.1 *In the perturbed auction $\Gamma^{\varepsilon,h}$, there exists a unique undominated* ex post *equilibrium in continuous strategies.*[7] *The equilibrium is given by (5) and (6).*

PROOF: The first order necessary conditions lead to a unique candidate profile (5) and (6) for Bayesian Nash equilibrium. I now argue that this strategy profile is indeed an *ex post* equilibrium.

Suppose that the realization of signals (s_1, s_2) is such that $\beta_1^\varepsilon(s_1) > \beta_2^\varepsilon(s_2)$, then bidder 1 wins the auction and pays $\beta_2^\varepsilon(s_2)$. Since the bidding strategies β_1^ε and β_2^ε are increasing functions, and ties occur at $\tilde{s}_1 = h(\tilde{s}_2)$, it follows that $s_1 > h(s_2)$. Since v is strictly increasing, bidder 1's ex post payoff is

$$\begin{aligned} &\varepsilon s_1 + (1-\varepsilon)v(s_1, s_2) - \beta_2^\varepsilon(s_2) \\ &= \varepsilon(s_1 - h(s_2)) + (1-\varepsilon)(v(s_1, s_2) - v(h(s_2), s_2)) \\ &> 0. \end{aligned}$$

[7] This equilibrium outcome is efficient in the perturbed auction. As we pointed out before, there are also discontinuous equilibria in the perturbed auction, but all those equilibria are inefficient.

Therefore, bidder 1 gets a positive surplus. Moreover, since she cannot affect the payment in a second-price auction, bidding $\beta_1^\varepsilon(s_1)$ is an ex post best response to $\beta_2^\varepsilon(s_2)$. On the other hand, bidder 2, who loses the auction and gets 0 payoff, cannot make a positive payoff by bidding larger than $\beta_1^\varepsilon(s_1)$, since

$$\begin{aligned}
&\varepsilon h(s_2) + (1-\varepsilon)v(s_1, s_2) - \beta_1^\varepsilon(s_1) \\
&= \varepsilon(h(s_2) - s_1) + (1-\varepsilon)(v(s_1, s_2) - v(s_1, h^{-1}(s_1))) \\
&< 0.
\end{aligned}$$

Therefore, bidding $\beta_2^\varepsilon(s_2)$ is an ex post best reply to $\beta_1^\varepsilon(s_1)$ for bidder 2. The case in which the signal realization (s_1, s_2) satisfies $\beta_1^\varepsilon(s_1) \leq \beta_2^\varepsilon(s_2)$ follows from a similar argument. □

Remark 3.2 Bikhchandani and Riley (1991) remark that in the pure common-value auction, the first-order conditions only pin down a single equation of two unknown bids. This in turn leads to the existence of a continuum of equilibria. In light of their discussion, we find that the presence of a private-value component gives another equation in the two unknown bids, which results in equilibrium uniqueness in the perturbed auction.

As ε goes to 0, the equilibrium $\beta^\varepsilon = (\beta_1^\varepsilon, \beta_2^\varepsilon)$ in the perturbed auction $\Gamma^{\varepsilon,h}$ converges to one of the equilibria in the common value auction Γ. That is, as $\varepsilon \to 0$, the bidding strategies (3) and (4) converges to

$$\beta_1(s_1) = v(s_1, h^{-1}(s_1)), \quad \text{and} \tag{7}$$

$$\beta_2(s_2) = v(h(s_2), s_2), \tag{8}$$

respectively. Therefore, in the common-value second-price auction, every ex post equilibrium that has the form (7) and (8) with $h \in \mathcal{H}$ is the limiting equilibrium of a sequence of perturbed "almost common-value" auctions. Moreover, each perturbed "almost common-value" auction has a unique continuous and undominated equilibrium. The result is summarized in the next proposition.

Proposition 3.3 *For any undominated ex post equilibrium β of the common-value second-price auction Γ satisfying (7) and (8) for some $h \in \mathcal{H}$, there is a sequence of perturbed "almost common-value" second-price auctions $\{\Gamma^{\varepsilon_k}\}$ such that each auction Γ^{ε_k} has a unique undominated and continuous Bayesian Nash equilibrium β^k, and the corresponding sequence of equilibria $\{\beta^k\}$ converges to β as ε^k goes to zero.*

Remark 3.4 Note that the above results hold under more general form of private-value perturbations. For example, instead of considering s_1 and $h(s_2)$ as bidders' private values, we can assume that bidder i's payoff is $\varepsilon^k w_{i,k}(s_i) + (1-\varepsilon^k)v(s_1, s_2)$, where $\varepsilon^k \to 0$ as $k \to \infty$, and bidder i's private value component $w_{i,k}(s_i)$ may vary with k. Suppose that for each $i = 1, 2$ and each $k \in \mathbb{N}$, $w_{i,k}$ is a strictly increasing and continuous function that maps $[0, 1]$ onto $[0, 1]$. For each $k \in \mathbb{N}$ and each $s_2 \in [0, 1]$, define $h_k(s_2) = {w_{1,k}}^{-1}(w_{2,k}(s_2))$. Following the steps that lead to equations (5) and (6), for each k, the equilibrium of the perturbed auction is characterized by

$$\beta_1^k(s_1) = \varepsilon^k w_{1,k}(s_1) + (1-\varepsilon^k)v(s_1, h_k^{-1}(s_1)),$$

$$\beta_2^k(s_2) = \varepsilon^k w_{2,k}(s_2) + (1-\varepsilon^k)v(h_k(s_2), s_2)).$$

If for each pair (s_1, s_2), $(w_{1,k}(s_1), w_{2,k}(s_2))$ converges to $(s_1, h(s_2))$, then the sequence of functions $\{h_k\}$ converges pointwise to h. It follows that as k goes to infinity, the sequence of equilibria $\{\beta^k\}$ converges to β, which is given by (7) and (8).[8]

As discussed in the introduction, the last result gives a negative answer to the question of equilibrium selection in pure common-value second-price auctions. In particular, each asymmetric equilibrium in the common-value auction can be justified by a particular perturbation of the model. If all possible perturbations are equally likely ex ante, then it is not quite clear why one equilibrium is more appealing than the others. Therefore, without embedding the current model into a larger context, analysis of common-value second-price auctions based on selecting a particular equilibrium is rather incomplete.

On the other hand, the class of perturbed auctions considered in this paper provides an explanation of the potential sources of the multiplicity. It identifies a one-to-one mapping from the class of perturbed auctions to the set of equilibria in the common-value auction. It is fair to argue that pure common-value auctions are rare in real world situations, the private-value component considered here can be interpreted as a reduced-form modeling of other factors that are relevant to bidders. The results in this paper suggest that, as bidders put more weight on their common-value components ($\varepsilon \to 0$), the unique efficient equilibrium in the "almost common-value" auction leads to a unique prediction of the bidding behavior in the limiting common-value model.

Remark 3.5 The same equilibrium selection results carry over to common-value auctions with more than two bidders. When there are three more bidders, the second-price auction and the English auction are no longer strategically equivalent. Yet similar argument can be applied to show that under either auction format, adding a private-value component to the common value has the power of selecting essentially any equilibrium. Instead of giving complete proofs of the results in the general n-bidder ($n \geq 3$) case, which are notationally involved, we provide an example with three bidders.

[8] I thank an anonymous referee for suggesting this kind of perturbations.

EXAMPLE 3.6 Consider the pure common-value auction with three symmetric bidders.[9] Assume that each bidder i's signal $\tilde{s}_i \in [0,1]$ and that the value and signals are strictly affiliated. Two auction formats are discussed successively.

1. In a second-price auction, there is a continuum of Bayesian Nash equilibria if the following condition is satisfied[10]

$$\hat{v}(s_1, s_2; s_3) = \hat{v}(s_1, s_3; s_2), \quad \forall s_1, s_2, s_3, \tag{9}$$

where $\hat{v}(s_i, s_j; s_k) = \mathbb{E}\left[V(\tilde{s}_i, \tilde{s}_j, \tilde{s}_k) | \tilde{s}_i = s_i, \tilde{s}_j = s_j, \tilde{s}_k \leq s_k\right]$, for $i \neq j \neq k$. Let $\bar{\mathcal{H}}$ be the set of strictly increasing functions that map $[0,1]$ onto $[0,1]$. Under condition (9), then for each pair of functions $h_2, h_3 \in \bar{\mathcal{H}}$, the following strategy profile is a Bayesian Nash equilibrium

$$\begin{aligned}
\beta_1(s_1) &= \hat{v}(s_1, h_2^{-1}(s_1); h_3^{-1}(s_1)), \\
\beta_2(s_2) &= \hat{v}(s_2, h_2(s_2); h_3^{-1}(h_2(s_2))), \\
\beta_3(s_3) &= \hat{v}(s_3, h_3(s_3); h_2^{-1}(h_3(s_3))).
\end{aligned}$$

Suppose the perturbed the values of the object to bidders satisfy

$$\begin{aligned}
\tilde{v}_1^{\varepsilon}(s_1, s_2, s_3) &= \varepsilon s_1 + (1-\varepsilon) v(s_1, s_2, s_3), \\
\tilde{v}_2^{\varepsilon}(s_1, s_2, s_3) &= \varepsilon h_2(s_2) + (1-\varepsilon) v(s_1, s_2, s_3), \\
\tilde{v}_3^{\varepsilon}(s_1, s_2, s_3) &= \varepsilon h_3(s_3) + (1-\varepsilon) v(s_1, s_2, s_3).
\end{aligned}$$

In the corresponding perturbed auction Γ^{ε}, the unique undominated and continuous Bayesian Nash equilibrium[11] is

$$\begin{aligned}
\beta_1^{\varepsilon}(s_1) &= \varepsilon s_1 + (1-\varepsilon)\hat{v}(s_1, h_2^{-1}(s_1); h_3^{-1}(s_1)), \\
\beta_2^{\varepsilon}(s_2) &= \varepsilon h_2(s_2) + (1-\varepsilon)\hat{v}(s_2, h_2(s_2); h_3^{-1}(h_2(s_2))), \\
\beta_3^{\varepsilon}(s_3) &= \varepsilon h_3(s_3) + (1-\varepsilon)\hat{v}(s_3, h_3(s_3); h_2^{-1}(h_3(s_3))).
\end{aligned}$$

Note that the strategy profile β^{ε} converges to β as ε goes to zero. Therefore, any equilibrium can be selected based on private-value perturbations.

2. In an English auction, there is always a continuum of *ex post* equilibria. In particular, when there are two bidders left, the situation is equivalent to a second-price auction with two-bidders with updated beliefs based on the dropout prices. Thus, the equilibrium multiplicity as well as the perturbation argument in the two-bidder case can be applied directly here.

[9] In this common-value environment, symmetry means that any permutation of bidders' signals does not change the expected value of the object conditional on the signals.

[10] The condition is identified by Bikhchandani and Riley (1991). They also provide a sufficient condition under which the second-price auction has a unique increasing and continuous equilibrium.

[11] The equilibrium is not *ex post* in the perturbed auction.

However, Bikhchandani and Riley (1991) also discover a larger class of equilibria in the English auction. We restrict attention to equilibrium selection here and omit the details of the description of those equilibria. Following closely the argument in the two-bidder case, It can be shown that for any continuous and undominated equilibrium β of the common-value English auction Δ in which the strategies do not depend directly on the dropout prices, there is a perturbed English auction Δ^ε which possesses a unique continuous and undominated *ex post* equilibrium β^ε. Again, β^ε converges to β as ε goes to zero.

4. ASYMMETRICALLY-INFORMED BIDDERS

In this section, we apply the payoff perturbation idea to study a common-value second-price auction with asymmetrically informed bidders.

Consider a common-value auction with one informed bidder and one uninformed bidder. Let bidder 1 be the informed bidder. Thus, bidder 2's signal s_2 does not generate any information about the value of the object. Write $v(s_1)$ as the expected value conditional on bidder 1's signal s_1 and assume v is strictly increasing in s_1.

The model with asymmetrically informed bidders under first-price auction format was introduced and studied by Engelbrecht-Wiggans, Milgrom and Weber (1983). Most notably, they find that in the unique equilibrium the uninformed bidder randomizes in such a way that the informed bidder behaves as if she were in a symmetric private-value first-price auction, and that the uninformed bidder obtains zero expected payoff.

However, the corresponding common-value second-price auction possesses a plethora of equilibria. For example, bidder 1, who is the informed bidder, has a weakly dominant strategy of bidding $\beta_1(s_1) = v(s_1)$ when her signal realization is s_1, and bidder 2, who is uninformed, can submit any bid b_2 that is weakly larger than $v(0)$. It is easy to see that any such strategy profile consists of an *ex post* equilibrium. Moreover, seller's revenue ranges from the lowest possible value $v(0)$ to the expected value $v(s_1)$ in different equilibria.

Abraham et al. (2012) consider the possibility of a noisy bidder whose bid follows a commonly known smooth distribution. They find that a unique equilibrium is selected, in which the uninformed bidder bids the lowest possible value $v(0)$.

Here we show that the same equilibrium can also be selected via private-value perturbation. In addition, it is shown that any equilibrium in this model can be selected by considering a particular form of the private-value component.

Fix the informed bidder's dominant strategy $\beta_1(s_1) = v(s_1)$ and consider perturbing the uninformed bidder's payoff. First, let $\underline{v}_2(s_1) = -\varepsilon + v(s_1)$ be bidder 2's value of the object for some $\varepsilon > 0$. Then it is clear that bidder 2 bids $v(0)$ in equilibrium, since she will end up with a non-positive payoff whenever she wins the auction. That is, an arbitrarily small payoff disadvantage of the uninformed bidder will always cause her to bid the minimum amount. Similarly, if bidder 2's value is $\bar{v}_2(s_1) = \varepsilon + v(s_1)$, then her equilibrium bid will be the highest possible expected value $v(\bar{s}_1)$.

More generally, let $g : [0, 1] \to [-\varepsilon, \varepsilon]$ be a continuous, strictly increasing and onto function. Thus, there is a unique $\hat{s}_1 \in [0, 1]$ such that $g(\hat{s}_1) = 0$ and $g(s_1) < 0$ for $s_1 < \hat{s}_1$, $g(s_1) > 0$ for $s_1 > \hat{s}_1$. Suppose bidder 2's expected value conditional on the realized signal s_1 is $\tilde{v}_2^g(s_1) = g(s_1) + v(s_1)$. Then bidder 2's optimal bid is $v(\hat{s}_1)$. To see this, first note that any bid b_2 smaller than $v(\hat{s}_1)$ is weakly dominated by $v(\hat{s}_1)$, since if b_2 is a winning bid then $v(\hat{s}_1)$ is also a winning bid which results in the same surplus to bidder 2, if b_2 is a losing bid then bidding $v(\hat{s}_1)$ results in at least zero payoff and sometimes a positive surplus (when $s_1 \in (v^{-1}(b_2), \hat{s}_1)$) to bidder 2. Similarly, any bid larger than $v(\hat{s}_1)$ is also weakly dominated by $v(\hat{s}_1)$. Therefore, the perturbed auction has a unique undominated equilibrium.

5. DISCONTINUOUS EQUILIBRIA IN SECOND PRICE AUCTIONS

The preceding analysis has focused on equilibria in continuous strategies. In fact, the common-value second-price auction also possesses equilibria in discontinuous strategies which may not even be monotone. In this section, we introduce another perturbation of the common-value auction following Abraham et al. (2012) and Hashimoto (2010) and provide a justification of our interest in continuous equilibria with a new robustness notion. In particular, we show that all those discontinuous equilibria are fragile to the presence of a noisy bid. To the contrary, all continuous equilibria studied in Section 2 and 3 are robust to this perturbation. The analysis here pertains to the two-bidder auction, but the results can be generalized to the n-bidder case.

Equilibria in discontinuous strategies can be constructed from any continuous equilibrium. For expositional simplicity, we consider the symmetric equilibrium $\beta = (\beta_1, \beta_2)$ where $\beta_i(s_i) = v(s_i, s_i)$ for $i = 1, 2$. Pick any $s', s'' \in (0, 1)$ with $s' < s''$. The following discontinuous strategies for the two bidders

$$\tilde{\beta}_1(s_1) = \begin{cases} v(s_1, s'') & \text{if } s_1 \in [s', s''] \\ v(s_1, s_1) & \text{otherwise} \end{cases}$$

$$\tilde{\beta}_2(s_2) = \begin{cases} v(s', s_2) & \text{if } s_2 \in [s', s''] \\ v(s_2, s_2) & \text{otherwise} \end{cases}$$

form an *ex post* equilibrium. Since s' and s'' are chosen arbitrarily, there is a continuum of equilibria in discontinuous strategies. Moreover, non-monotone equilibria can be constructed from the above discontinuous equilibrium by twisting the bids for

$s_1, s_2 \in [s', s'']$. For example, pick any $\hat{s} \in (s', s'')$, it is easy to see that the strategy profile

$$\hat{\beta}_1(s_1) = \begin{cases} v(s'', s'') & \text{if } s_1 \in [s', \hat{s}) \\ v(s_1, s'') & \text{if } s_1 \in [\hat{s}, s''] \\ v(s_1, s_1) & \text{otherwise} \end{cases}$$

$$\hat{\beta}_2(s_2) = \tilde{\beta}_2(s_2)$$

is also an *ex post* equilibrium. In fact, for bidder 1 with signal $s_1 \in [s', s'']$, any bid in the interval $(v(s', s''), v(s'', s''))$ can be supported in an equilibrium. Likewise, for bidder 2 with signal $s_2 \in [s', s'']$, any bid strictly between $v(s', s')$ and $v(s', s'')$ can be an equilibrium bid.

The intuition for the fragility of the discontinuous equilibria to noisy bids is simple. While those discontinuous equilibria are undominated, they are in some sense close to being dominated. In any such discontinuous equilibrium, one bidder sometimes bids aggressively as she believes that her opponent would submit lower bids. The presence of a noisy bid could cause her to pay more than the value of the object and hence make her more cautious. Therefore, no equilibrium in discontinuous strategies is the limit of equilibria in perturbed auctions with noisy bids.

Formally, suppose each bidder i believes that with probability ε an additional bidder enters the auction and bids randomly according to a strictly positive density function g over the set of all possible bids. Let $\mathrm{BR}^{\varepsilon,g}(\beta_{-i}) : [0,1] \to \mathbb{R}_+$ denote the best response correspondence for bidder i to her opponent's strategy β_{-i}.

Definition 5.1 An equilibrium $\beta = (\beta_1, \beta_2)$ of the common-value second-price auction is said to be *robust to noisy bids* if for each $i = 1, 2$, for each $s_i \in [0, 1]$,

$$\lim_{\varepsilon \to 0} d\left(\beta_i(s_i), \mathrm{BR}^{\varepsilon,g}(\beta_{-i})(s_i)\right) = 0,$$

where $d(x, S) = \inf\{|x - y| : y \in S\}$ is the distance from a point x to a set S.

To see that equilibria in discontinuous strategies are not robust, first consider the strategy profile $(\tilde{\beta}_1, \tilde{\beta}_2)$ constructed above. For each $s_1 \in (s', s'']$, define

$$k(s_1) \equiv \mathbb{E}_{S_2}[v(\tilde{s}_1, \tilde{s}_2) | \tilde{s}_2 \leq s'', \tilde{s}_1 = s_1]$$

and

$$l(s_1) \equiv \max\left\{k(s_1), v(s', s'')\right\}.$$

Since v is strictly increasing in s_2, there exists $\delta > 0$ such that $\tilde{\beta}_1(s_1) - k(s_1) > \delta$ for each $s_1 \in (s', s'']$. The next result shows that $\tilde{\beta}_1(s_1)$ is bounded away from the set of best responses $\mathrm{BR}^{\varepsilon,g}(\tilde{\beta}_2)(s_1)$.

Proposition 5.2 *Given the perturbation g, for any $\varepsilon > 0$ and any signal $s_1 \in (s', s'']$, bidder 1 with signal s_1 strictly prefers to bid $l(s_1) < \tilde{\beta}_1(s_1)$ rather than $\tilde{\beta}_1(s_1)$, given bidder 2's strategy $\tilde{\beta}_2$.*

Proof: Conditional on the event that the noisy bid is less than bidder 2's bid, bidder 1 is indifferent between $v(s', s'')$ and $\tilde{\beta}_1(s_1)$, since conditional on winning with either bid, bidder 1's payment is determined by $\tilde{\beta}_2(s_2)$.

On the other hand, if the noisy bid is larger than bidder 2's bid, then conditional on winning, bidder 1 has to pay the noisy bid. Since bidder 2 never submits bids between $v(s', s'')$ and $v(s'', s'')$ under $\tilde{\beta}_2$, any payment $p \in (l(s_1), \tilde{\beta}_1(s_1))$ indicates that the expected payoff of bidder 1 is negative, i.e.,

$$\mathbb{E}_{S_2}[v(\tilde{s}_1, \tilde{s}_2)|\tilde{s}_2 \leq s'', \tilde{s}_1 = s_1] - p = k(s_1) - p < k(s_1) - l(s_1) \leq 0.$$

Since the density of the noisy bid g has full support by assumption, the noisy bids lies in $(l(s_1), \tilde{\beta}_1(s_1))$ with positive probability. Therefore, it is never optimal for bidder 1 with signal $s_1 \in (s', s'']$ to bid above $l(s_1)$ in the presence of noisy bids. □

The above proposition implies that $(\tilde{\beta}_1, \tilde{\beta}_2)$ does not meet our robustness criterion. The source of fragility comes from the discontinuities in both bidders' strategies, as the proof indicates. Therefore, the proof can easily be generalized to show that all discontinuous equilibria constructed along the line of $\tilde{\beta}$ or $\hat{\beta}$ are not robust to noisy bids. In contrast, the next result demonstrates the robustness of the class of equilibria identified by Milgrom (1981).

Proposition 5.3 *For each strictly increasing and onto function $h \in \mathcal{H}$, the equilibrium of the second-price auction (β_1, β_2), where $\beta_1(s_1) = v(s_1, h^{-1}(s_1))$ and $\beta_2(s_2) = v(h(s_2), s_2)$, is robust to noisy bids.*

Proof: Fix a function $h \in \mathcal{H}$ and consider the strategy profile (β_1, β_2) where $\beta_1(s_1) = v(s_1, h^{-1}(s_1))$ and $\beta_2(s_2) = v(h(s_2), s_2)$. We need to show that for each $i = 1, 2$ and each $s_i \in [0, 1]$,

$$\lim_{\varepsilon \to 0} d(\beta_i(s_i), \mathrm{BR}^{\varepsilon, g}(\beta_{-i})(s_i)) = 0.$$

First consider bidder 1 who receives any signal $s_1 \in [0, 1]$. We show that the above limit is zero in two steps.

Step 1: We show that for any $\varepsilon > 0$, and any $b \in \mathrm{BR}^{\varepsilon, g}(\beta_2)(s_1)$, $b \leq \beta_1(s_1)$. That is, the set of best responses for bidder 1 with signal s_1 is below $\beta_1(s_1)$. This can be seen from the payoff function of bidder 1 in the presence of the noisy bid

$$\begin{aligned}
(10)\qquad &\Pi(b, s_1; \beta_2, \varepsilon, g)\\
&= \int_0^{\beta_2^{-1}(b)} ((1-\varepsilon) + \varepsilon G(\beta_2(s_2)))\,[v(s_1, s_2) - \beta_2(s_2)]\, f_2(s_2|s_1) ds_2\\
&\quad + \varepsilon \int_0^{\beta_2^{-1}(b)} \left[\int_{\beta_2(s_2)}^{b} [v(s_1, s_2) - x] g(x) dx\right] f_2(s_2|s_1) ds_2.
\end{aligned}$$

Note that the the first term on the right-hand side of equation (10) is concave and its unique maximizer is $b = \beta_1(s_1)$. Now suppose there exists $b \in \mathrm{BR}^{\varepsilon,g}(\beta_2)(s_1)$ such that $b > \beta_1(s_1)$, since $\beta_2^{-1}(\beta_1(s_1)) = h^{-1}(s_1)$ and h is strictly increasing and onto, there exists $\bar{s}_2 > h^{-1}(s_1)$ such that $\beta_2(\bar{s}_2) = b$. Then we have

$$\begin{aligned}
(11)\qquad &\Pi(b, s_1; \beta_2, \varepsilon, g) - \Pi(\beta_1(s_1), s_1; \beta_2, \varepsilon, g)\\
&< \varepsilon \int_0^{\beta_2^{-1}(b)} \left[\int_{\beta_2(s_2)}^{b} [v(s_1, s_2) - x] g(x) dx\right] f_2(s_2|s_1) ds_2\\
&\quad - \varepsilon \int_0^{\beta_2^{-1}(\beta_1(s_1))} \left[\int_{\beta_2(s_2)}^{\beta_1(s_1)} [v(s_1, s_2) - x] g(x) dx\right] f_2(s_2|s_1) ds_2\\
&= \varepsilon \int_{h^{-1}(s_1)}^{\bar{s}_2} \left[\int_{\beta_2(s_2)}^{\beta_2(\bar{s}_2)} [v(s_1, s_2) - x] g(x) dx\right] f_2(s_2|s_1) ds_2\\
&\quad + \varepsilon \int_0^{h^{-1}(s_1)} \left[\int_{\beta_1(s_1)}^{\beta_2(\bar{s}_2)} [v(s_1, s_2) - x] g(x) dx\right] f_2(s_2|s_1) ds_2.
\end{aligned}$$

Note that (i) for each $s_2 \in [h^{-1}(s_1), \bar{s}_2]$, we have $h(s_2) \geq s_1$ and

$$\begin{aligned}
(12)\qquad &\int_{\beta_2(s_2)}^{\beta_2(\bar{s}_2)} [v(s_1, s_2) - x] g(x) dx\\
&< [v(s_1, s_2) - \beta_2(s_2)] \cdot [G(\beta_2(\bar{s}_2)) - G(\beta_2(s_2))]\\
&= [v(s_1, s_2) - v(h(s_2), s_2)] \cdot [G(\beta_2(\bar{s}_2)) - G(\beta_2(s_2))]\\
&\leq 0;
\end{aligned}$$

and (ii) for each $s_2 \in [0, h^{-1}(s_1)]$, we have

$$\begin{aligned}
(13)\qquad &\int_{\beta_1(s_1)}^{\beta_2(\bar{s}_2)} [v(s_1, s_2) - x] g(x) dx\\
&< [v(s_1, s_2) - \beta_1(s_1)] \cdot [G(\beta_2(\bar{s}_2)) - G(\beta_1(s_1))]\\
&= [v(s_1, s_2) - v(s_1, h^{-1}(s_1))] \cdot [G(\beta_2(\bar{s}_2)) - G(\beta_1(s_1))]\\
&\leq 0.
\end{aligned}$$

Therefore, combining (11), (12) and (13) gives

$$\Pi(b, s_1; \beta_2, \varepsilon, g) - \Pi(\beta_1(s_1), s_1; \beta_2, \varepsilon, g) < 0,$$

which contradicts the presumption that b is a best response to β_2 for bidder 1 with signal s_1 in the perturbed auction.

Step 2: We show that for any $\eta > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $b \geq \beta_1(s_1) - \eta$ for any $b \in \mathrm{BR}^{\varepsilon,g}(\beta_2)(s_1)$. That is, $\beta_1(s_1)$ is approximately optimal as ε goes to 0.

By contradiction, suppose there exists $\eta > 0$ such that for any $\varepsilon > 0$, there exists $b \in \mathrm{BR}^{\varepsilon,g}(\beta_2)(s_1)$ such that $b < \beta_1(s_1) - \eta$. Since we have

$$\begin{aligned} &\Pi(\beta_1(s_1) - \eta, s_1; \beta_2, \varepsilon, g) - \Pi(b, s_1; \beta_2, \varepsilon, g) \\ &> (1-\varepsilon)\Delta + \varepsilon \int_0^{\beta_2^{-1}(\beta_1(s_1)-\eta)} \left[\int_{\beta_2(s_2)}^{\beta_1(s_1)-\eta} [v(s_1, s_2) - x] g(x) dx \right] f_2(s_2|s_1) ds_2 \\ &\quad - \varepsilon \int_0^{\beta_2^{-1}(b)} \left[\int_{\beta_2(s_2)}^{b} [v(s_1, s_2) - x] g(x) dx \right] f_2(s_2|s_1) ds_2 \\ &\geq (1-\varepsilon)\Delta - 4\varepsilon v(1,1), \end{aligned} \tag{14}$$

where $\Delta \equiv \int_b^{\beta_1(s_1)-\eta} [v(s_1, s_2) - v(h(s_2), s_2)] f_2(s_2|s_1) ds_2$ is strictly positive and is independent of ε. Therefore, for small enough ε, (14) implies that

$$\Pi(\beta_1(s_1) - \eta, s_1; \beta_2, \varepsilon, g) - \Pi(b, s_1; \beta_2, \varepsilon, g) > 0,$$

which is a contradiction.

The argument for bidder 2 with signal s_2 is similar and is omitted. □

We conclude with a discussion of the connection between our robustness notion and Selten's trembling-hand perfection (Selten (1975)).[12] Both notions share the feature of using full support strategies to select certain equilibria. In normal form games, perfection rules out equilibria in weakly dominated strategies. In a second-price auction (a Bayesian game), our robustness notion excludes equilibria in discontinuous strategies. However, there is an important distinction between the two notions. While trembling-hand perfection considers totally mixed strategies through the perturbation of strategy sets, our robustness notion simply requires each bidder to believe that "any rival bid is possible" even if her opponent's strategy does not have full support.

The reason why we do not pursue strategy perturbations is that in our common-value environment, each bidder's strategy provides additional information about the

[12] See also Simon and Stinchcombe (1995) for notions of perfection in infinite normal-form games.

value of the object to her opponents, and strategy perturbations would inevitably confound each bidder's inference problem. The introduction of noisy bids, on the other hand, (i) does not provide direct information about the valuation, (ii) does not interfere with bidders' inference problem which is a crucial element in a common-value model, and (iii) is yet powerful enough to exclude all the "seemingly dominated" discontinuous equilibria. Thus, we think that perturbation by noisy bids with full support seems to be closer in spirit to the notion of perfection in Selten (1975) comparing with strategy perturbations.

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